Numerical Linear Algebra

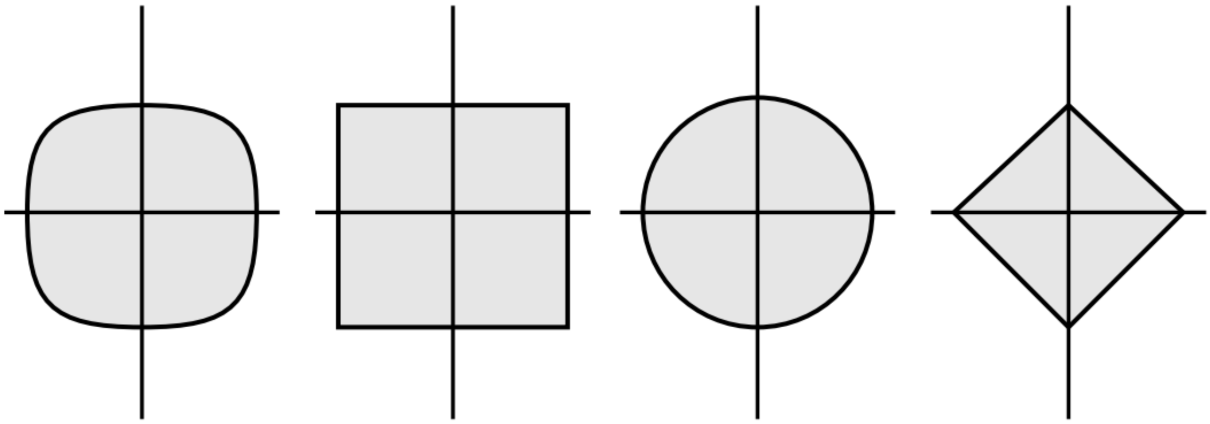
Norms

1. Vector Norms

A *norm* is a function $\|\cdot\| : \mathbb{R}^m \to \mathbb{R}$ that assigns a real-valued length to each vector that satisfy the following three conditions.

$$
\begin{aligned}
&(1)\ \|x\| \geq 0, \text{and } \|x\| = 0 \text{ only if } x = 0,\\
&(2)\ \|x + y\| \leq \|x\| + \|y\|,\\
&(3)\ \|\alpha x\| = |\alpha|\,\|x\|.
\end{aligned}
$$

Examples:



$$
\begin{aligned}
||x||_1 &= \sum_{i=1}^{m} |x_i|, \\
||x||_2 &= \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \sqrt{x^T x}, \\
||x||_\infty &= \max_{1 \le i \le m} |x_i|, \\
||x||_p &= \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p} \quad (1 \le p < \infty).
\end{aligned}
$$

2. Matrix Norms Induced by Vector Norms

$A \in \mathbb{R}^{m \times n}$, the matrix norm can be defined:

$$
||A||_{(m,n)} = \sup_{x \in \mathbb{R}^n \ x \neq 0} \frac{||Ax||_{(m)}}{||x||_{(n)}} = \sup_{x \in \mathbb{R}^n \ ||x||_{(n)} = 1} ||Ax||_{(m)}.
$$

The p-Norm of a diagonal Matrix:

$$
D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_m \end{bmatrix}
$$

then $||D||_p = \max_{1 \le i \le m} |d_i|$.

The 1-Norm of a Matrix:

$A \in \mathbb{R}^{m \times n}$, then $||A||_1$ id equal to the "maximum column sum" of A.

$$
||Ax||_1 = ||\sum_{j=1}^{n} x_j a_j||_1 \le \sum_{j=1}^{n} |x_j| \, ||a_j||_1 \le \max_{1 \le j \le n} ||a_j||_1.
$$

$$
||A||_1 = \max_{1 \le j \le n} ||a_j||_1.
$$

The ∞-Norm of a Matrix:

$$||A||_\infty = \max_{1 \le j \le m} ||a_j^T||_1,$$

where a_j^T denotes the jth row of A.

3. Cauchy-Schwarz and Holder Inequalities

Let p and q satisfy $1/p + 1/q = 1$, with $1 \le p, q \le \infty$. Then the *Holder inequality* states that, for any vectors x and y,

$$|x^T y| \le ||x||_p ||y||_q.$$

4. Bounding $||AB||$ in an Induced Matrix Norm

$$||AB||_{(l,n)} \le ||A||_{(l,m)} ||B||_{(m,n)}.$$

5. General Matrix Norms

Frobenius norm:

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

$$||AB||_F^2 = ||A||_F^2 ||B||_F^2$$

Theorem 1. For any $A \in \mathbb{R}^{m \times n}$ and unitary $Q \in \mathbb{R}^{m \times m}$, we have

$$||QA||_2 = ||A||_2, \quad ||QA||_F = ||A||_F.$$