Numerical Linear Algebra

Singular Value Decomposition

1. Geometric Observation

The SVD is motivated by the following geometric fact:

The image of the unit sphere under any $m \times n$ matrix is a hyperellipse.

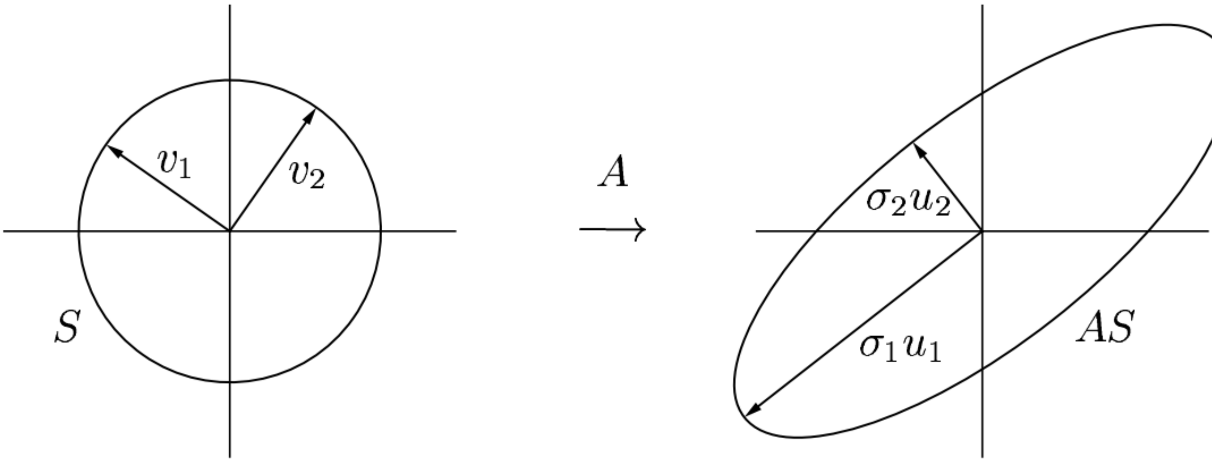


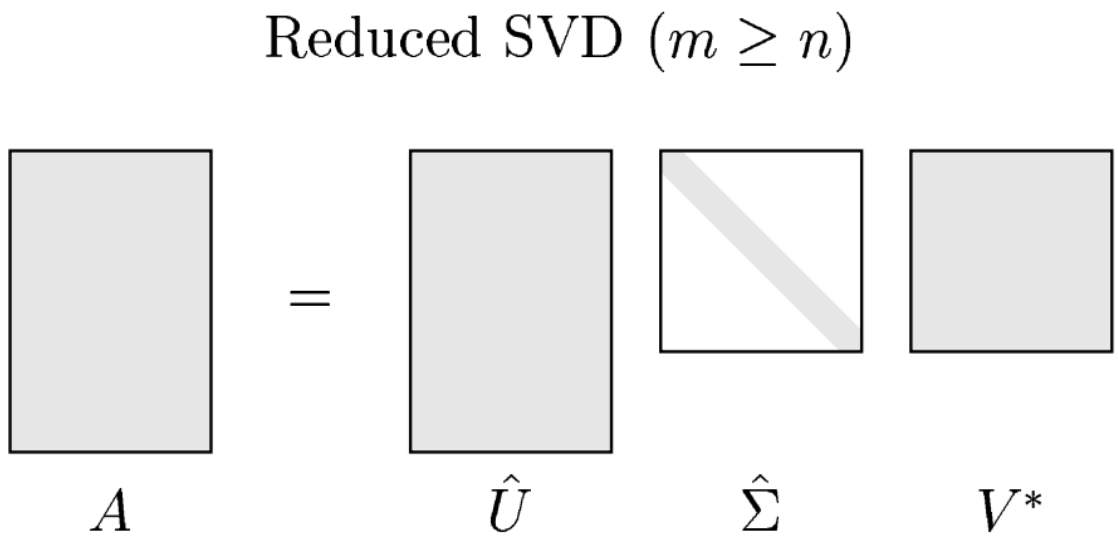
Figure 4.1. *SVD of a* 2×2 *matrix.*

We define the n*singular values* of A: $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the lengths of the n principal semiaxes of AS, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Define the n*left singular vectors* of A are the unit vectors $\{u_1, u_2, \ldots, u_n\}$ oriented in the directions of the principal semiaxes of AS. Define the n*right singular vectors* of A are the unit vectors $\{v_1, v_2, \ldots, v_n\} \in S$ that are the preimages of the principal semiaxes of AS, so $Av_j = \sigma_j u_j$.

2. Reduced SVD

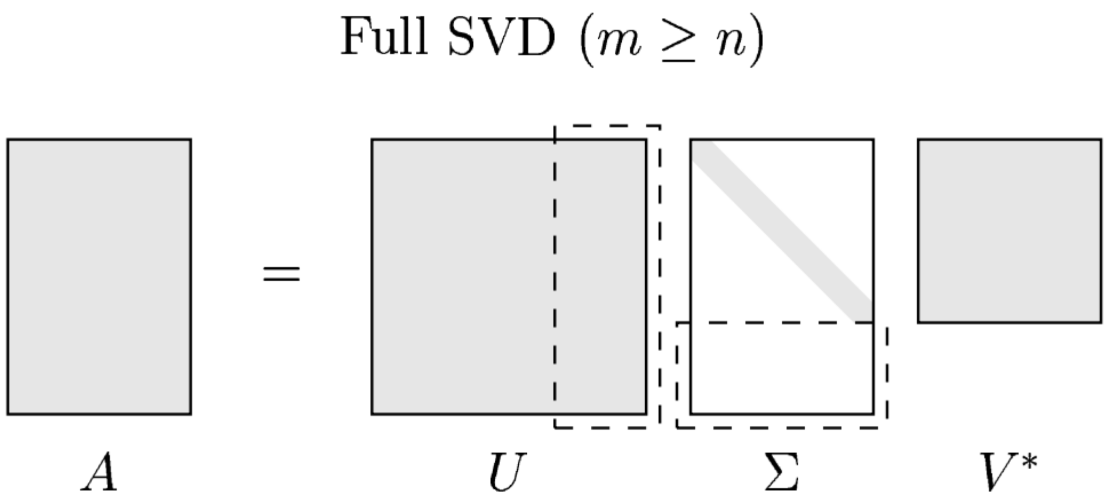
$A \in \mathbb{R}^{m \times n} (m > n)$ and we assume A has full rank n:

$$A = \hat{U}\hat{\Sigma}V^T.$$



3. Full SVD

$$A = U\Sigma V^T.$$



4. Formal Definition

Given $A \in \mathbb{R}^{m \times n}$, a *singular value decomposition* (SVD) of A is a factorization

$$A = U\Sigma V^T$$

where

$$\begin{aligned} U &\in \mathbb{R}^{m\times m} && \text{is unitary,} \\ V &\in \mathbb{R}^{n\times n} && \text{is unitary,} \\ \Sigma &\in \mathbb{R}^{m\times n} && \text{is diagonal.} \end{aligned}$$

Theorem 1. Every matrix $A \in \mathbb{R}^{m\times n}$ has a singular value decomposition. Furthermore, the singular values $\{\sigma_j\}$ are uniquely determined, and, if A is square and the σ_j are distinct, the left and right singular vectors $\{u_j\}$ and $\{v_j\}$ are uniquely determined up to complex signs.

Matrix properties via the SVD

Theorem 2. The rank of A is r, the number of nonzero singular values.

Theorem 3. $\text{range}(A) = \langle u_1, \ldots, u_r \rangle$ and $\text{null}(A) = \langle v_{r+1}, \ldots, v_n \rangle$.

Theorem 4. $||A||_2 = \sigma_1$ and $||A||_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.

Theorem 5. The nonzero singular values of A are the square roots of the nonzero eigenvalues of A^TA or AA^T.

Theorem 6. For $A \in \mathbb{R}^{m\times m}$, $|\det(A)| = \prod_{i=1}^{m} \sigma_i$.

Low-Rank Approximations

Theorem 7. A is the sum of r rank-one matrices:

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T.$$

Theorem 8. For any ν with $0 \le \nu \le r$, define

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^T;$$

if $\nu = p = \min\{m, n\}$, define $\sigma_{\nu+1} = 1$. Then

$$\|A - A_\nu\|_2 = \inf_{\substack{B \in \mathbb{R}^{m \times n} \\ \text{rank}(B) \le \nu}} \|A - B\|_2 = \sigma_{\nu+1}.$$

Theorem 9. For any ν with $0 \le \nu \le r$, the matrix A_ν also satisfies

$$\|A - A_\nu\|_F = \inf_{\substack{B \in \mathbb{R}^{m \times n} \\ \text{rank}(B) \le \nu}} \|A - B\|_F = \sqrt{\sigma_{\nu+1}^2 + \cdots + \sigma_r^2}.$$